WEAK HARDY SPACES

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Abstract. We provide a careful treatment of the weak Hardy spaces $H^{p,\infty}(\mathbf{R}^n)$ for all indices $0 < p < \infty$. The study of these spaces presents differences from the study of the Hardy-Lorentz spaces $H^{p,q}(\mathbf{R}^n)$ for $q < \infty$, due to the lack of a good dense subspace of them. We obtain several properties of weak Hardy spaces and we discuss a new square function characterization for them, obtained by He [16].

Contents

1. Introduction

The impact of the theory of Hardy spaces in the last forty years has been significant. The higher dimensional Euclidean theory of Hardy spaces was developed by Fefferman and Stein [10] who proved a variety of characterizations for them. A deep atomic decomposition characterization of these spaces was given by Coifman [3] in dimension one and by Latter [18] in higher dimensions. The treatments of Hardy spaces in Lu [19], García-Cuerva and Rubio de Francia [11], Grafakos [13], Stein [24], and Triebel [25] cover the main aspects of their classical theory in the Euclidean setting. Among hundreds of references on this topic, the works of Coifman, and Weiss [4], Macías, and Segovia [22], Duong, and Yan [7], Han, Müller and Yang [15], Hu, Yang, and Zhou [17] contain powerful extensions of the theory of Hardy spaces to the setting of spaces of homogeneous type. A new type of Hardy space, called Herz-type Hardy space was introduced by Lu and Yang [21] to measure the localization fine-tuned on cubical shells centered at the origin.

In this work we provide a careful treatment of the weak Hardy space H^p (henceforth $H^{p,\infty}$) on $\mathbf{R}^n$ for $0 < p < \infty$. This is defined as the space of all bounded tempered

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distributions whose Poisson maximal function lies in weak L^p (or $L^{p,\infty}$). The study of these spaces presents certain crucial differences from the study of the Hardy-Lorentz spaces $H^{p,q}$ when $q < \infty$, due to the fact that they lack a good dense subspace of smooth functions such as the Schwartz class; this is explained in Theorem 6 and it was also observed by Fefferman and Soria [9]. As a consequence, certain results concerning these spaces cannot be proved by restricting attention to the Schwartz class. In this article we bypass this difficulty to obtain several maximal characterizations of weak Hardy spaces working directly from the definition and we prove an interpolation result for these spaces working with general bounded distributions.

It is well known that function spaces can be characterized in terms of Littlewood-Paley square function expressions. Such characterizations are given in Ding, Lu, Xue [6], Lu, Yang [20], Peetre [23], and Triebel [25], but we discuss here a new square function characterization of $H^{p,\infty}$ in terms of the Littlewood-Paley operators, a result that was recently obtained by He [16].

Fefferman, Riviere, and Sagher [8], Fefferman and Soria [9], Alvarez [2], Abu-Shammala and Torchinsky [1] have obtained a variety of results concerning the weak H^p spaces. Fefferman, Riviere, and Sagher [8] have studied interpolation between the Hardy-Lorentz spaces $H^{p,q}$. Fefferman and Soria [9] carefully investigated the space $H^{1,\infty}$, and they provided its atomic decomposition. Alvarez [2] provided the atomic decomposition of the spaces $H^{p,\infty}$ and she also studied the action of singular integrals on them. In [8], Fefferman, Riviere and Sagher obtained the spaces $H^{p,q}$ as an intermediate interpolation space of Hardy spaces, but their proof was only given for Schwartz functions which are not dense in $H^{p,\infty}$, and thus their proof contains an incomplete deduction in the case $q = \infty$. He [16] overcomes the technical issues arising from the lack of density of Schwartz functions via a Calderón-Zygmund type decomposition for general weak H^p distributions. Some results in the literature of weak Hardy spaces are based on the interpolation result in [8] and possibly on the assumption that locally integrable functions are dense in this space. Although the latter is unknown as of this writing, the former is possible and is explained here.

Our exposition builds the theory of weak Hardy spaces, starting from the classical definition of the Poisson maximal function. We discuss various maximal characterizations of these spaces and we state an interpolation theorem for $H^{p,\infty}$ from initial strong H^{p_0} and H^{p_1} estimates with $p_0 < p < p_1$. Using this interpolation result, the second author [16] has obtained a new square function characterization for the spaces $H^{p,\infty}$, which is presented here without proof. This characterization is based on a singular integral estimate for vector-valued weak H^p spaces. For this reason, we develop the theory of weak Hardy spaces in the vector-valued setting.

2. Relevant background

To introduce the vector-valued weak Hardy spaces we need a sequence of definitions given in this section. We denote by ℓ^2 the space $\ell^2(\mathbf{Z})$ of all square-integrable sequences and by $\ell^2(L)$ the finite-dimensional space of all sequences of length $L \in \mathbf{Z}^+$ with the ℓ^2 norm. We say that a sequence of distributions $\{f_j\}_j$ lies in $\mathcal{S}'(\mathbf{R}^n, \ell^2)$ if there are constants $C, M > 0$ such that for every $\varphi \in \mathcal{S}(\mathbf{R}^n)$ we have

$$\big\|\{\langle f_j, \varphi\rangle\}_j\big\|_{\ell^2} = \Big(\sum_j |\langle f_j, \varphi\rangle|^2\Big)^{1/2} \le C \sum_{|\alpha|,|\beta|\le M} \sup_{y\in\mathbf{R}^n} |y^\beta \partial^\alpha \varphi(y)|.$$

And this sequence of distributions $\vec{f} = \{f_j\}_j$ is called bounded if for any $\varphi \in \mathcal{S}(\mathbf{R}^n)$ we have

$$\big\|\{\varphi * f_j\}_j\big\|_{\ell^2} = \Big(\sum_j |\varphi * f_j|^2\Big)^{1/2} \in L^\infty(\mathbf{R}^n). \tag{1}$$

Let $a, b > 0$ and let Φ be a Schwartz function on $\mathbf{R}^n$.

Definition 1. *For a sequence $\vec{f} = \{f_j\}_{j\in\mathbf{Z}}$ of tempered distributions on $\mathbf{R}^n$ we define the smooth maximal function of $\vec{f}$ with respect to Φ as*

$$\boldsymbol{M}(\vec{f};\Phi)(x) = \sup_{t>0} \big\|\{(\Phi_t * f_j)(x)\}_j\big\|_{\ell^2}.$$

We define the nontangential maximal function with aperture a of $\vec{f}$ with respect to Φ as

$$\boldsymbol{M}_a^*(\vec{f};\Phi)(x) = \sup_{t>0} \sup_{\substack{y\in\mathbf{R}^n\\|y-x|\le at}} \big\|\{(\Phi_t * f_j)(y)\}_j\big\|_{\ell^2}.$$

We also define the auxiliary maximal function

$$\boldsymbol{M}_b^{**}(\vec{f};\Phi)(x) = \sup_{t>0} \sup_{y\in\mathbf{R}^n} \frac{\big\|\{(\Phi_t * f_j)(x-y)\}_j\big\|_{\ell^2}}{(1+t^{-1}|y|)^b}.$$

For a fixed positive integer N we define the grand maximal function of $\vec{f}$ as

$$\mathcal{M}_N(\vec{f}) = \sup_{\varphi\in\mathcal{F}_N} \boldsymbol{M}_1^*(\vec{f};\varphi), \tag{2}$$

where

$$\mathcal{F}_N = \Big\{\varphi \in \mathcal{S}(\mathbf{R}^n):\ \mathfrak{N}_N(\varphi) \le 1\Big\}, \tag{3}$$

and

$$\mathfrak{N}_N(\varphi) = \int_{\mathbf{R}^n} (1+|x|)^N \sum_{|\alpha|\le N+1} |\partial^\alpha \varphi(x)|\, dx$$

is the "norm" of φ. More generally, we define the "norm" of φ adapted to the pair $(x_0, R) \in \mathbf{R}^n \times \mathbf{R}^+$ by setting

$$\mathfrak{N}_N(\varphi; x_0, R) = \int_{\mathbf{R}^n} \Big(1 + \Big|\frac{x-x_0}{R}\Big|\Big)^N \sum_{|\alpha|\le N+1} R^{|\alpha|} |\partial^\alpha \varphi(x)|\, dx.$$

Note that $\mathfrak{N}_N(\varphi; 0, 1) = \mathfrak{N}_N(\varphi)$.

If the function Φ is the Poisson kernel, then the maximal functions $\boldsymbol{M}(\vec{f};\Phi)$, $\boldsymbol{M}_a^*(\vec{f};\Phi)$, and $\boldsymbol{M}_b^{**}(\vec{f};\Phi)$ are well defined for sequences of bounded tempered distributions $\vec{f}=\{f_j\}_j$ in view of (1).

We note that the following simple inequalities

$$\boldsymbol{M}(\vec{f};\Phi)\le \boldsymbol{M}_a^*(\vec{f};\Phi)\le (1+a)^b\boldsymbol{M}_b^{**}(\vec{f};\Phi) \tag{4}$$

are valid. We now define the vector-valued Hardy space $H^{p,\infty}(\mathbf{R}^n,\ell^2)$.

Definition 2. *Let $\vec{f}=\{f_j\}_j$ be a sequence of bounded tempered distributions on $\mathbf{R}^n$ and let $0<p<\infty$. We say that $\vec{f}$ lies in the vector-valued weak Hardy space $H^{p,\infty}(\mathbf{R}^n,\ell^2)$ vector-valued Hardy space if the Poisson maximal function*

$$\boldsymbol{M}(\vec{f};P)(x)=\sup_{t>0}\big\|\{(P_t*f_j)(x)\}_j\big\|_{\ell^2}$$

lies in $L^{p,\infty}(\mathbf{R}^n)$. If this is the case, we set

$$\big\|\vec{f}\big\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2)}=\big\|\boldsymbol{M}(\vec{f};P)\big\|_{L^{p,\infty}(\mathbf{R}^n)}=\Big\|\sup_{\varepsilon>0}\Big(\sum_j|P_\varepsilon*f_j|^2\Big)^{\frac12}\Big\|_{L^{p,\infty}(\mathbf{R}^n)}.$$

The next theorem provides a characterization of $H^{p,\infty}$ in terms of different maximal functions.

Theorem 1. *Let $0<p<\infty$. Then the following statements are valid:*
(a) There exists a Schwartz function Φ with integral 1 and a constant C_1 such that

$$\big\|\boldsymbol{M}(\vec{f};\Phi)\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)}\le C_1\big\|\vec{f}\big\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2)} \tag{5}$$

for every sequence $\vec{f}=\{f_j\}_j$ of tempered distributions.
(b) For every $a>0$ and Φ in $\mathcal{S}(\mathbf{R}^n)$ there exists a constant $C_2(n,p,a,\Phi)$ such that

$$\big\|\boldsymbol{M}_a^*(\vec{f};\Phi)\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)}\le C_2(n,p,a,\Phi)\big\|\boldsymbol{M}(\vec{f};\Phi)\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)} \tag{6}$$

for every sequence $\vec{f}=\{f_j\}_j$ of tempered distributions.
(c) For every $a>0$, $b>n/p$, and Φ in $\mathcal{S}(\mathbf{R}^n)$ there exists a constant $C_3(n,p,a,b,\Phi)$ such that

$$\big\|\boldsymbol{M}_b^{**}(\vec{f};\Phi)\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)}\le C_3(n,p,a,b,\Phi)\big\|\boldsymbol{M}_a^*(\vec{f};\Phi)\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)} \tag{7}$$

for every sequence $\vec{f}=\{f_j\}_j$ of tempered distributions.
(d) For every $b>0$ and Φ in $\mathcal{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n}\Phi(x)\,dx\neq 0$ there exists a constant $C_4(b,\Phi)$ such that if $N=[b]+1$ we have

$$\big\|\mathcal{M}_N(\vec{f})\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)}\le C_4(b,\Phi)\big\|\boldsymbol{M}_b^{**}(\vec{f};\Phi)\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)} \tag{8}$$

for every sequence $\vec{f}=\{f_j\}_j$ of tempered distributions.
(e) For every positive integer N there exists a constant $C_5(n,N)$ such that every sequence $\vec{f}=\{f_j\}_j$ of tempered distributions that satisfies $\big\|\mathcal{M}_N(\vec{f})\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)}<\infty$ is bounded and satisfies

$$\big\|\vec{f}\big\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2)}\le C_5(n,N)\big\|\mathcal{M}_N(\vec{f})\big\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2)}, \tag{9}$$

that is, it lies in the Hardy space $H^{p,\infty}(\mathbf{R}^n, \ell^2)$.

We conclude that for bounded distributions $\vec{f} = \{f_j\}$ the following equivalence of quasi-norms holds

$$\big\|\mathcal{M}_N(\vec{f})\big\|_{L^{p,\infty}} \approx \big\|M_b^{**}(\vec{f};\Phi)\big\|_{L^{p,\infty}} \approx \big\|M_a^{*}(\vec{f};\Phi)\big\|_{L^{p,\infty}} \approx \big\|M(\vec{f};\Phi)\big\|_{L^{p,\infty}}$$

with constants that depend only on Φ, a, n, p, and all the preceding quasi-norms are also equivalent with $\|\vec{f}\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2)}$.

There is an alternative characterization of the weak H^p quasi-norm via the weak L^p quasi-norm of the associated square function. As usual, we denote by

$$\Delta_j(f) = \Delta_j^{\Psi}(f) = \Psi_{2^{-j}} * f$$

the Littlewood-Paley operator of f, where $\Psi_t(x) = t^{-n}\Psi(x/t)$.

Theorem 2. *([16]) Let Ψ be a radial Schwartz function on $\mathbf{R}^n$ whose Fourier transform is nonnegative, supported in $\frac{1}{2}+\frac{1}{10} \le |\xi| \le 2-\frac{1}{10}$, and satisfies (40). Let Δ_j be the Littlewood–Paley operators associated with Ψ and let $0 < p < \infty$. Then there exists a constant $C = C_{n,p,\Psi}$ such that for all $f \in H^p(\mathbf{R}^n)$ we have*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p,\infty}} \le C\big\|f\big\|_{H^{p,\infty}}. \tag{10}$$

Conversely, suppose that a tempered distribution f satisfies

$$\Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p,\infty}} < \infty. \tag{11}$$

Then there exists a unique polynomial $Q(x)$ such that $f-Q$ lies in $H^{p,\infty}$ and satisfies the estimate

$$\frac{1}{C}\big\|f-Q\big\|_{H^{p,\infty}} \le \Big\|\Big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p,\infty}}. \tag{12}$$

The following version of the classical Fefferman-Stein vector-valued inequality [10] is useful. This result for upper Boyd indices less than infinity is contained in [5] (page 85). A self-contained proof of the following result is contained in [16].

Proposition 1. *If $1 < p, q < \infty$, then for all sequences of functions $\{f_j\}_j$ in $L^{p,\infty}(\ell^q)$ we have*

$$\big\|\|\{M(f_j)\}\|_{\ell^q}\big\|_{L^{p,\infty}} \le C_{p,q}\,\big\|\|\{f_j\}\|_{\ell^q}\big\|_{L^{p,\infty}},$$

where M is the Hardy-Littlewood maximal function.

3. The proof of Theorem 1

The proof of this theorem is based on the following lemma whose proof can be found in [13].

Lemma 1. *Let $m \in \mathbf{Z}^+$ and let Φ in $\mathcal{S}(\mathbf{R}^n)$ satisfy $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$. Then there exists a constant $C_0(\Phi, m)$ such that for any Ψ in $\mathcal{S}(\mathbf{R}^n)$, there are Schwartz functions $\Theta^{(s)}$, $0 \le s \le 1$, with the properties*

$$\Psi(x) = \int_0^1 (\Theta^{(s)} * \Phi_s)(x)\,ds \tag{13}$$

and

$$\int_{\mathbf{R}^n} (1+|x|)^m |\Theta^{(s)}(x)|\,dx \le C_0(\Phi, m)\, s^m\, \mathfrak{N}_m(\Psi). \tag{14}$$

We now prove Theorem 1.

Proof. (a) Consider the function $\psi(s)$ defined on the interval $[1,\infty)$ as follows:

$$\psi(s) = \frac{e}{\pi}\,\frac{1}{s}\,e^{-\frac{\sqrt{2}}{2}(s-1)^{\frac14}} \sin\left(\frac{\sqrt{2}}{2}(s-1)^{\frac14}\right). \tag{15}$$

Clearly $\psi(s)$ decays faster than any negative power of s and satisfies

$$\int_1^\infty s^k\,\psi(s)\,ds = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, 2, 3, \dots . \end{cases} \tag{16}$$

We now define the function

$$\Phi(x) = \int_1^\infty \psi(s) P_s(x)\,ds\,, \tag{17}$$

where P_s is the Poisson kernel. Note that the double integral

$$\int_{\mathbf{R}^n} \int_1^\infty \frac{s}{(s^2+|x|^2)^{\frac{n+1}{2}}}\, s^{-N}\,ds\,dx$$

converges and so it follows from (16) and (17) that $\int_{\mathbf{R}^n} \Phi(x)\,dx = 1$. Furthermore, we have that

$$\widehat{\Phi}(\xi) = \int_1^\infty \psi(s)\widehat{P_s}(\xi)\,ds = \int_1^\infty \psi(s) e^{-2\pi s|\xi|}\,ds$$

using that $\widehat{P_s}(\xi) = e^{-2\pi s|\xi|}$. This function is rapidly decreasing as $|\xi| \to \infty$ and the same is true for all the derivatives

$$\partial^\alpha \widehat{\Phi}(\xi) = \int_1^\infty \psi(s) \partial_\xi^\alpha \left(e^{-2\pi s|\xi|}\right) ds\,. \tag{18}$$

Moreover, the function $\widehat{\Phi}$ is clearly smooth on $\mathbf{R}^n \setminus \{0\}$ and we will show that it also smooth at the origin. Notice that for all multiindices α we have

$$\partial_\xi^\alpha (e^{-2\pi s|\xi|}) = s^{|\alpha|} p_\alpha(\xi) |\xi|^{-m_\alpha} e^{-2\pi s|\xi|}$$

for some $m_\alpha \in \mathbf{Z}^+$ and some polynomial $p_\alpha(\xi)$. By Taylor's theorem, for some function $v(s, |\xi|)$ with $0 \le v(s, |\xi|) \le 2\pi s|\xi|$, we have

$$e^{-2\pi s|\xi|} = \sum_{k=0}^{L} (-2\pi)^k \frac{|\xi|^k}{k!} s^k + \frac{(-2\pi s|\xi|)^{L+1}}{(L+1)!} e^{-v(s,|\xi|)}\,.$$

Choosing $L > m_\alpha$ gives

$$\partial_\xi^\alpha(e^{-2\pi s|\xi|}) = \sum_{k=0}^{L}(-2\pi)^k\frac{|\xi|^k}{k!}s^{k+|\alpha|}\frac{p_\alpha(\xi)}{|\xi|^{m_\alpha}} + s^{|\alpha|}\frac{p_\alpha(\xi)}{|\xi|^{m_\alpha}}\frac{(-2\pi s|\xi|)^{L+1}}{(L+1)!}e^{-v(s,|\xi|)},$$

which inserted in (18) and in view of (16), yields that when $|\alpha| > 0$, the derivative $\partial^\alpha\widehat{\Phi}(\xi)$ tends to zero as $\xi \to 0$ and when $\alpha = 0$, $\widehat{\Phi}(\xi) \to 1$ as $\xi \to 0$. We conclude that $\widehat{\Phi}$ is continuously differentiable, and hence smooth at the origin, hence it lies in the Schwartz class, and thus so does Φ.

Finally, we have the estimate

$$\begin{aligned}
M(\vec{f};\Phi)(x) &= \sup_{t>0}\Big(\sum_j |(\Phi_t * f_j)(x)|^2\Big)^{1/2} \\
&= \sup_{t>0}\Big(\sum_j \Big|\int_1^\infty \psi(s)(P_{ts} * f_j)(x)\, ds\Big|^2\Big)^{1/2} \\
&\le \Big(\int_1^\infty |\psi(s)|\, ds\Big)^{1/2} \sup_{t>0}\Big(\sum_j \int_1^\infty |(P_{ts} * f_j)(x)|^2\, |\psi(s)|\, ds\Big)^{1/2} \\
&\le \Big(\int_1^\infty |\psi(s)|\, ds\Big)^{1/2} \Big(\int_1^\infty \sup_{t>0}\sum_j |(P_{ts} * f_j)(x)|^2\, |\psi(s)|\, ds\Big)^{1/2} \\
&\le \int_1^\infty |\psi(s)|\, ds\, M(\vec{f};P)(x)\,,
\end{aligned}$$

and the required conclusion follows since $\int_1^\infty |\psi(s)|\, ds \le C_1$. We have actually obtained the pointwise estimate $M(\vec{f};\Phi) \le C_1\, M(\vec{f};P)$ which clearly implies (5).

(b) We present the proof only in the case when $a = 1$ since the case of general $a > 0$ is similar. We derive (6) as a consequence of the estimate

$$\big\|M_1^*(\vec{f};\Phi)\big\|_{L^{p,\infty}}^p \le C_2''(n,p,\Phi)^p\big\|M(\vec{f};\Phi)\big\|_{L^{p,\infty}}^p + \frac{1}{2}\big\|M_1^*(\vec{f};\Phi)\big\|_{L^{p,\infty}}^p\,, \tag{19}$$

which requires a priori knowledge of the fact that $\|M_1^*(\vec{f};\Phi)\|_{L^{p,\infty}} < \infty$. This presents a significant hurdle that needs to be overcome by an approximation. For this reason we introduce a family of maximal functions $M_1^*(\vec{f};\Phi)^{\varepsilon,N}$ for $0 \le \varepsilon, N < \infty$ such that $\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\|_{L^p} < \infty$ and such that $M_1^*(\vec{f};\Phi)^{\varepsilon,N} \uparrow M_1^*(\vec{f};\Phi)$ as $\varepsilon \downarrow 0$ and we prove (19) with $M_1^*(\vec{f};\Phi)^{\varepsilon,N}$ in place of $M_1^*(\vec{f};\Phi)^{\varepsilon,N}$, i.e., we prove

$$\big\|\, M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}}^p \le C_2'(n,p,\Phi,N)^p\big\|M(\vec{f};\Phi)\big\|_{L^{p,\infty}}^p + \frac{1}{2}\big\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}}^p\,, \tag{20}$$

where there is an additional dependence on N in the constant $C_2'(n,p,\Phi,N)$, but there is no dependence on ε. The $M_1^*(\vec{f};\Phi)^{\varepsilon,N}$ are defined as follows: for a bounded distribution $\vec{f}$ in $\mathcal{S}'(\mathbf{R}^n,\ell^2)$ such that $M(\vec{f};\Phi)\in L^p$ we define

$$M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)=\sup_{0<t<\frac{1}{\varepsilon}}\sup_{|y-x|<t}\bigg(\sum_j\big|(\Phi_t*f_j)(y)\big|^2\bigg)^{1/2}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\,.$$

We first show that $M_1^*(\vec{f};\Phi)^{\varepsilon,N}$ lies in $L^p(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ if N is large enough depending on $\vec{f}$. Indeed, using that $(\Phi_t*f_j)(x)=\langle f_j,\Phi_t(x-\cdot)\rangle$ and the fact that $\vec{f}$ is in $\mathcal{S}'(\mathbf{R}^n,\ell^2)$, we obtain constants $C=C_{\vec{f}}$ and $m=m_{\vec{f}}$, $m>n$ such that:

$$\begin{aligned}\bigg(\sum_j\big|(\Phi_t*f_j)(y)\big|^2\bigg)^{\frac12}\le\;& C\sum_{|\gamma|\le m,|\beta|\le m}\sup_{w\in\mathbf{R}^n}|w^\gamma(\partial^\beta\Phi_t)(y-w)|\\ \le\;& C\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|y|^m+|z|^m)|(\partial^\beta\Phi_t)(z)|\\ \le\;& C\,(1+|y|^m)\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|z|^m)|(\partial^\beta\Phi_t)(z)|\\ \le\;& C\,\frac{(1+|y|^m)}{\min(t^n,t^{n+m})}\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|z|^m)|(\partial^\beta\Phi)(z/t)|\\ \le\;& C\,\frac{(1+|y|)^m}{\min(t^n,t^{n+m})}(1+t^m)\sum_{|\beta|\le m}\sup_{z\in\mathbf{R}^n}(1+|z/t|^m)|(\partial^\beta\Phi)(z/t)|\\ \le\;& C_{\vec{f},\Phi}(1+\varepsilon|y|)^m\varepsilon^{-m}(1+t^m)(t^{-n}+t^{-n-m})\,.\end{aligned}$$

Multiplying by $(\frac{t}{t+\varepsilon})^N(1+\varepsilon|y|)^{-N}$ for some $0<t<\frac{1}{\varepsilon}$ and $|y-x|<t$ yields

$$\bigg(\sum_j\big|(\Phi_t*f_j)(y)\big|^2\bigg)^{1/2}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\le C_{\vec{f},\Phi}\frac{\varepsilon^{-m}(1+\varepsilon^{-m})(\varepsilon^{n-N}+\varepsilon^{n+m-N})}{(1+\varepsilon|y|)^{N-m}}\,,$$

and using that $1+\varepsilon|y|\ge\frac12(1+\varepsilon|x|)$, we obtain for some $C''(\vec{f},\Phi,\varepsilon,n,m,N)<\infty$,

$$M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\le\frac{C''(\vec{f},\Phi,\varepsilon,n,m,N)}{(1+\varepsilon|x|)^{N-m}}\,.$$

Taking $N>m+n/p$, we have that $M_1^*(\vec{f};\Phi)^{\varepsilon,N}$ lies in $L^{p,\infty}(\mathbf{R}^n)$. This choice of N depends on m and hence on the sequence of distributions $\vec{f}=\{f_j\}_j$.

We now introduce functions

$$U(\vec{f};\Phi)^{\varepsilon,N}(x)=\sup_{0<t<\frac{1}{\varepsilon}}\sup_{|y-x|<t}t\bigg(\sum_j\big|\nabla(\Phi_t*f_j)(y)\big|^2\bigg)^{1/2}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}$$

and

$$V(\vec{f};\Phi)^{\varepsilon,N}(x)=\sup_{0<t<\frac{1}{\varepsilon}}\sup_{y\in\mathbf{R}^n}\bigg[\sum_j\big|(\Phi_t*f_j)(y)\big|^2\bigg]^{\frac12}\Big[\frac{t}{t+\varepsilon}\Big]^N\frac{1}{(1+\varepsilon|y|)^N}\Big[\frac{t}{t+|x-y|}\Big]^{[\frac{2n}{p}]+1}.$$

We need the norm estimate

$$\big\|V(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}} \le C(n)^{\frac{2}{p}}\big\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}} \tag{21}$$

and the pointwise estimate

$$U(\vec{f};\Phi)^{\varepsilon,N} \le A(n,p,\Phi,N)\, V(\vec{f};\Phi)^{\varepsilon,N}\,, \tag{22}$$

where

$$A(\Phi,N,n,p) = 2^{[\frac{2n}{p}]+1}\sum_{j=1}^{n} C_0(\partial_j\Phi, N+[\tfrac{2n}{p}]+1)\,\mathfrak{N}_{N+[\frac{2n}{p}]+1}(\partial_j\Phi)\,.$$

To prove (21) we observe that when $z\in B(y,t)\subseteq B(x,|x-y|+t)$ we have

$$\Big(\sum_j \big|(\Phi_t*f_j)(y)\big|^2\Big)^{\frac{1}{2}}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N} \le M_1^*(f;\Phi)^{\varepsilon,N}(z)\,,$$

from which it follows that for any $y\in\mathbf{R}^n$,

$$\begin{aligned}
&\Big(\sum_j \big|(\Phi_t*f_j)(y)\big|^2\Big)^{\frac{1}{2}}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y|)^N}\\
&\le \bigg(\frac{1}{|B(y,t)|}\int_{B(y,t)}\big[M_1^*(\vec{f};\Phi)^{\varepsilon,N}(z)\big]^{\frac{p}{2}}\,dz\bigg)^{\frac{2}{p}}\\
&\le \bigg(\frac{|x-y|+t}{t}\bigg)^{\frac{2n}{p}}\bigg(\frac{1}{|B(x,|x-y|+t)|}\int_{B(x,|x-y|+t)}\big[M_1^*(\vec{f};\Phi)^{\varepsilon,N}(z)\big]^{\frac{p}{2}}\,dz\bigg)^{\frac{2}{p}}\\
&\le \bigg(\frac{|x-y|+t}{t}\bigg)^{[\frac{2n}{p}]+1} M\Big(\big[M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big]^{\frac{p}{2}}\Big)^{\frac{2}{p}}(x)\,.
\end{aligned}$$

We now use the boundedness of the Hardy–Littlewood maximal operator M from $L^{2,\infty}$ to $L^{2,\infty}$ to obtain (21) as follows:

$$\begin{aligned}
\big\|V(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}} &= \Big\|M\big((M_1^*(\vec{f};\Phi)^{\varepsilon,N})^{\frac{p}{2}}\big)^{\frac{2}{p}}\Big\|_{L^{2,\infty}}\\
&= \Big\|M\big((M_1^*(\vec{f};\Phi)^{\varepsilon,N})^{\frac{p}{2}}\big)\Big\|_{L^{2,\infty}}^{\frac{2}{p}}\\
&\le C(n)^{\frac{2}{p}}\Big\|(M_1^*(\vec{f};\Phi)^{\varepsilon,N})^{\frac{p}{2}}\Big\|_{L^{2,\infty}}^{\frac{2}{p}}\\
&= C(n)^{\frac{2}{p}}\big\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}}
\end{aligned}$$

In proving (22), we may assume that Φ has integral 1; otherwise we can multiply Φ by a suitable constant to arrange for this to happen. We note that for each $x\in\mathbf{R}^n$ we have

$$t\big\|\nabla(\Phi_t*\vec{f}\,)(x)\big\|_{\ell^2} = \big\|(\nabla\Phi)_t*\vec{f}\,(x)\big\|_{\ell^2} \le \sqrt{n}\sum_{j=1}^{n}\big\|(\partial_j\Phi)_t*\vec{f}\,(x)\big\|_{\ell^2}\,,$$

and it suffices to work with each partial derivative $\partial_j\Phi$. Using Lemma 1 we write

$$\partial_j\Phi = \int_0^1 \Theta^{(s)} * \Phi_s \, ds$$

for suitable Schwartz functions $\Theta^{(s)}$. Fix $x \in \mathbf{R}^n$, $t > 0$, and y with $|y-x| < t < 1/\varepsilon$. Then we have

$$\begin{aligned}
&\big\|\big((\partial_j\Phi)_t * \vec{f}\big)(y)\big\|_{\ell^2}\Big(\frac{t}{t+\varepsilon}\Big)^N \frac{1}{(1+\varepsilon|y|)^N}\\
&= \Big(\frac{t}{t+\varepsilon}\Big)^N \frac{1}{(1+\varepsilon|y|)^N}\bigg\|\int_0^1 \big((\Theta^{(s)})_t * \Phi_{st} * \vec{f}\big)(y)\, ds\bigg\|_{\ell^2} \\
&\le \Big(\frac{t}{t+\varepsilon}\Big)^N \int_0^1\int_{\mathbf{R}^n} t^{-n}\big|\Theta^{(s)}(t^{-1}z)\big| \,\frac{\big\|\big(\Phi_{st} * \vec{f}\big)(y-z)\big\|_{\ell^2}}{(1+\varepsilon|y|)^N}\, dz\ ds\,.
\end{aligned} \tag{23}$$

Inserting the factor 1 written as

$$\bigg(\frac{ts}{ts+|x-(y-z)|}\bigg)^{[\frac{2n}{p}]+1}\Big(\frac{ts}{ts+\varepsilon}\Big)^N\bigg(\frac{ts+|x-(y-z)|}{ts}\bigg)^{[\frac{2n}{p}]+1}\Big(\frac{ts+\varepsilon}{ts}\Big)^N$$

in the preceding z-integral and using that

$$\frac{1}{(1+\varepsilon|y|)^N} \le \frac{(1+\varepsilon|z|)^N}{(1+\varepsilon|y-z|)^N}$$

and the fact that $|x-y| < t < 1/\varepsilon$, we obtain the estimate

$$\begin{aligned}
&\Big(\frac{t}{t+\varepsilon}\Big)^N \int_0^1\int_{\mathbf{R}^n} t^{-n}\big|\Theta^{(s)}(t^{-1}z)\big| \,\frac{\big\|\big(\Phi_{st} * \vec{f}\big)(y-z)\big\|_{\ell^2}}{(1+\varepsilon|y|)^N}\, dz\, ds\\
&\le\ V(\vec{f};\Phi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbf{R}^n}(1+\varepsilon|z|)^N\bigg(\frac{ts+|x-(y-z)|}{ts}\bigg)^{[\frac{2n}{p}]+1} t^{-n}\big|\Theta^{(s)}(t^{-1}z)\big|\, dz\,\frac{ds}{s^N}\\
&\le\ V(\vec{f};\Phi)^{\varepsilon,N}(x)\int_0^1\int_{\mathbf{R}^n} s^{-[\frac{2n}{p}]-1-N}(1+\varepsilon t|z|)^N(s+1+|z|)^{[\frac{2n}{p}]+1}\big|\Theta^{(s)}(z)\big|\, dz\, ds\\
&\le\ 2^{[\frac{2n}{p}]+1}\, C_0(\partial_j\Phi, N+[\tfrac{2n}{p}]+1)\,\mathfrak{N}_{N+[\frac{2n}{p}]+1}(\partial_j\Phi)\, V(\vec{f};\Phi)^{\varepsilon,N}(x)
\end{aligned}$$

in view of conclusion (14) of Lemma 1. Combining this estimate with (23), we deduce (22). Estimates (21) and (22) together yield

$$\big\|U(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}} \le C(n)^{\frac{2}{p}}\, A(n,p,\Phi,N)\big\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}}\,. \tag{24}$$

We now set

$$E_\varepsilon = \big\{x\in\mathbf{R}^n:\ U(\vec{f};\Phi)^{\varepsilon,N}(x) \le K\, M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\big\}$$

for some constant K to be determined shortly. With $A = A(n,p,\Phi,N)$ we have

$$
\begin{aligned}
\big\|M_1^*(\vec f;\Phi)^{\varepsilon,N}\big\|^p_{L^{p,\infty}((E_\varepsilon)^c)} &\le \frac{1}{K^p}\big\|U(\vec f;\Phi)^{\varepsilon,N}\big\|^p_{L^{p,\infty}((E_\varepsilon)^c)} \\
&\le \frac{1}{K^p}\big\|U(\vec f;\Phi)^{\varepsilon,N}\big\|^p_{L^{p,\infty}} \\
&\le \frac{C(n)^2\,A^p}{K^p}\big\|M_1^*(\vec f;\Phi)^{\varepsilon,N}\big\|^p_{L^{p,\infty}} \\
&\le \frac{1}{2}\big\|M_1^*(\vec f;\Phi)^{\varepsilon,N}\big\|^p_{L^{p,\infty}}\,,
\end{aligned} \tag{25}
$$

provided we choose K such that $K^p = 2\,C(n)^p\,A(n,p,\Phi,N)^p$. Obviously K is a function of n,p,Φ,N and in particular depends on N.

It remains to estimate the weak $L^{p,\infty}$ quasi-norm of $M_1^*(f;\Phi)^{\varepsilon,N}$ over the set E_ε. We claim that the following pointwise estimate is valid:

$$
M_1^*(\vec f;\Phi)^{\varepsilon,N}(x) \le 4\,C'(n,N,K)^{\frac1q}\left[M\big(M(\vec f;\Phi)^q\big)(x)\right]^{\frac1q} \tag{26}
$$

for any $x \in E_\varepsilon$ and $0 < q < \infty$ and some constant $C'(n,N,K)$, where M is the Hardy-Littlewood maximal operator. For the proof of (26) we cite [13] in the scalar case, but we indicate below why the proof also holds in the vector-valued setting.

To prove (26) we fix $x \in E_\varepsilon$ and we also fix y such that $|y-x| < t$.

By the definition of $M_1^*(f;\Phi)^{\varepsilon,N}(x)$ there exists a point $(y_0,t) \in \mathbf{R}^{n+1}_+$ such that $|x-y_0| < t < \frac{1}{\varepsilon}$ and

$$
\big\|(\Phi_t * \vec f\,)(y_0)\big\|_{\ell^2}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|y_0|)^N} \ge \frac12 M_1^*(\vec f;\Phi)^{\varepsilon,N}(x)\,. \tag{27}
$$

By the definitions of E_ε and $U(\vec f;\Phi)^{\varepsilon,N}$, for any $x \in E_\varepsilon$ we have

$$
t\big\|\nabla(\Phi_t * \vec f\,)(\xi)\big\|_{\ell^2}\Big(\frac{t}{t+\varepsilon}\Big)^N\frac{1}{(1+\varepsilon|\xi|)^N} \le K\,M_1^*(\vec f;\Phi)^{\varepsilon,N}(x) \tag{28}
$$

for all ξ satisfying $|\xi - x| < t < \frac{1}{\varepsilon}$. It follows from (27) and (28) that

$$
t\big\|\nabla(\Phi_t * \vec f\,)(\xi)\big\| \le 2K\big\|(\Phi_t * \vec f\,)(y_0)\big\|_{\ell^2}\left(\frac{1+\varepsilon|\xi|}{1+\varepsilon|y_0|}\right)^N \tag{29}
$$

for all ξ satisfying $|\xi - x| < t < \frac{1}{\varepsilon}$. We let z be such that $|z-x| < t$ and $|z-y_0| < t$. Applying the mean value theorem and using (29), we obtain, for some ξ between y_0 and z,

$$
\begin{aligned}
\big\|(\Phi_t * \vec f\,)(z) - (\Phi_t * \vec f\,)(y_0)\big\|_{\ell^2} &= \big\|\nabla(\Phi_t * \vec f\,)(\xi)\big\|_{\ell^2}\,|z-y_0| \\
&\le \frac{2K}{t}\big\|(\Phi_t * \vec f\,)(\xi)\big\|_{\ell^2}\left(\frac{1+\varepsilon|\xi|}{1+\varepsilon|y_0|}\right)^N |z-y_0| \\
&\le \frac{2^{N+1}K}{t}\big\|(\Phi_t * \vec f\,)(y_0)\big\|_{\ell^2}\,|z-y_0| \\
&\le \frac12\big\|(\Phi_t * \vec f\,)(y_0)\big\|_{\ell^2}\,,
\end{aligned}
$$

provided z also satisfies $|z-y_0|<2^{-N-2}K^{-1}t$ in addition to $|z-x|<t$. Therefore, for z satisfying $|z-y_0|<2^{-N-2}K^{-1}t$ and $|z-x|<t$ we have

$$\big\|(\Phi_t*\vec{f})(z)\big\|_{\ell^2}\ge\frac{1}{2}\big\|(\Phi_t*\vec{f})(y_0)\big\|_{\ell^2}\ge\frac{1}{4}M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\,,$$

where the last inequality uses (27). Thus we have

$$\begin{aligned}M\big(M(\vec{f};\Phi)^q\big)(x)\;&\ge\;\frac{1}{|B(x,t)|}\int_{B(x,t)}\big[M(\vec{f};\Phi)(w)\big]^q\,dw\\&\ge\;\frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(y_0,2^{-N-2}K^{-1}t)}\big[M(\vec{f};\Phi)(w)\big]^q\,dw\\&\ge\;\frac{1}{|B(x,t)|}\int_{B(x,t)\cap B(y_0,2^{-N-2}K^{-1}t)}\frac{1}{4^q}\big[M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\big]^q\,dw\\&\ge\;\frac{|B(x,t)\cap B(y_0,2^{-N-2}K^{-1}t)|}{|B(x,t)|}\frac{1}{4^q}\big[M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\big]^q\\&\ge\;C'(n,N,K)^{-1}4^{-q}\big[M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\big]^q\,,\end{aligned}$$

where we used the simple geometric fact that if $|x-y_0|\le t$ and $\delta>0$, then

$$\frac{|B(x,t)\cap B(y_0,\delta t)|}{|B(x,t)|}\ge c_{n,\delta}>0\,,$$

the minimum of this constant being obtained when $|x-y_0|=t$. This proves (26).

Taking $q=p/2$ and applying the boundedness of the Hardy–Littlewood maximal operator on $L^{2,\infty}$ yields

$$\big\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}(E_\varepsilon)}\le C_2'(n,p,\Phi,N)\big\|M(\vec{f};\Phi)\big\|_{L^{p,\infty}}\,. \tag{30}$$

Combining this estimate with (25), we finally prove (20).

Recalling the fact (obtained earlier) that $\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\|_{L^{p,\infty}}<\infty$, we deduce from (20) that

$$\big\|M_1^*(\vec{f};\Phi)^{\varepsilon,N}\big\|_{L^{p,\infty}}\le 2^{\frac{1}{p}}C_2'(n,p,\Phi,N)\big\|M(\vec{f};\Phi)\big\|_{L^{p,\infty}}\,. \tag{31}$$

The preceding constant depends on $\vec{f}$ but is independent of ε. Notice that

$$M_1^*(\vec{f};\Phi)^{\varepsilon,N}(x)\ge\frac{2^{-N}}{(1+\varepsilon|x|)^N}\sup_{0<t<1/\varepsilon}\Big(\frac{t}{t+\varepsilon}\Big)^N\sup_{|y-x|<t}\big\|(\Phi_t*\vec{f})(y)\big\|_{\ell^2}$$

and that the previous expression on the right increases to

$$2^{-N}M_1^*(\vec{f};\Phi)(x)$$

as $\varepsilon\downarrow 0$. Since the constant in (31) does not depend on ε, an application of the Lebesgue monotone convergence theorem yields

$$\|M_1^*(\vec{f};\Phi)\|_{L^{p,\infty}}\le 2^{N+\frac{1}{p}}C_2'(n,p,\Phi,N)\|M(\vec{f};\Phi)\|_{L^{p,\infty}}\,. \tag{32}$$

The problem with this estimate is that the finite constant $2^NC_2'(n,p,\Phi,N)$ depends on N and thus on $\vec{f}$. However, we have managed to show that under the assumption $\|M(\vec{f};\Phi)\|_{L^{p,\infty}}<\infty$, one must necessarily have $\|M_1^*(\vec{f};\Phi)\|_{L^{p,\infty}}<\infty$.

Keeping this significant observation in mind, we repeat the preceding argument from the point where the functions $U(\vec{f};\phi)^{\varepsilon,N}$ and $V(\vec{f};\phi)^{\varepsilon,N}$ are introduced, setting $\varepsilon = N = 0$. Then we arrive to (19) with a constant $C_2''(n,p,\Phi) = C_2'(n,p,\Phi,0)$ which is independent of N and thus of $\vec{f}$. We conclude the validity of (6) with $C_2(n,p,1,\Phi) = 2^{1/p}C_2''(n,p,\Phi)$ when $a = 1$. An analogous constant is obtained for different values of $a > 0$.

(c) Let $B(x,R)$ denote a ball centered at x with radius R. Recall that

$$M_b^{**}(\vec{f};\Phi)(x) = \sup_{t>0}\sup_{y\in\mathbf{R}^n} \frac{\|(\Phi_t * \vec{f})(x-y)\|_{\ell^2}}{\left(\frac{|y|}{t}+1\right)^b}.$$

It follows from the definition

$$M_a^*(\vec{f};\Phi)(z) = \sup_{t>0}\sup_{|w-z|<at} \|(\Phi_t * \vec{f})(w)\|_{\ell^2}$$

that

$$\|(\Phi_t * \vec{f})(x-y)\|_{\ell^2} \le M_a^*(\vec{f};\Phi)(z) \qquad \text{if } \ z \in B(x-y,at)\,.$$

But the ball $B(x-y,at)$ is contained in the ball $B(x,|y|+at)$; hence it follows that

$$\begin{aligned}
\|(\Phi_t * \vec{f})(x-y)\|_{\ell^2}^{\frac{n}{b}} &\le \frac{1}{|B(x-y,at)|}\int_{B(x-y,at)} M_a^*(\vec{f};\Phi)(z)^{\frac{n}{b}}\,dz \\
&\le \frac{1}{|B(x-y,at)|}\int_{B(x,|y|+at)} M_a^*(\vec{f};\Phi)(z)^{\frac{n}{b}}\,dz \\
&\le \left(\frac{|y|+at}{at}\right)^n M\big(M_a^*(\vec{f};\Phi)^{\frac{n}{b}}\big)(x) \\
&\le \max(1,a^{-n})\left(\frac{|y|}{t}+1\right)^n M\big(M_a^*(\vec{f};\Phi)^{\frac{n}{b}}\big)(x)\,,
\end{aligned}$$

from which we conclude that for all $x \in \mathbf{R}^n$ we have

$$M_b^{**}(\vec{f};\Phi)(x) \le \max(1,a^{-b})\left\{M\big(M_a^*(\vec{f};\Phi)^{\frac{n}{b}}\big)(x)\right\}^{\frac{b}{n}}.$$

We now take $L^{p,\infty}$ norms on both sides of this inequality and using the fact that $pb/n > 1$ and the boundedness of the Hardy–Littlewood maximal operator M from $L^{pb/n,\infty}$ to itself, we obtain the required conclusion (7).

(d) In proving (d) we may replace b by the integer $b_0 = [b]+1$. Let Φ be a Schwartz function with integral equal to 1. Applying Lemma 1 with $m = b_0$, we write any function φ in $\mathcal{F}_N$ as

$$\varphi(y) = \int_0^1 (\Theta^{(s)} * \Phi_s)(y)\,ds$$

for some choice of Schwartz functions $\Theta^{(s)}$. Then we have

$$\varphi_t(y) = \int_0^1 ((\Theta^{(s)})_t * \Phi_{ts})(y)\,ds$$

for all $t>0$. Fix $x\in\mathbf{R}^n$. Then for y in $B(x,t)$ we have

$$\begin{aligned}\|(\varphi_t*\vec{f})(y)\|_{\ell^2}\le &\int_0^1\int_{\mathbf{R}^n}|(\Theta^{(s)})_t(z)|\,\|(\Phi_{ts}*\vec{f})(y-z)\|_{\ell^2}\,dz\;ds\\ \le &\int_0^1\int_{\mathbf{R}^n}|(\Theta^{(s)})_t(z)|\,M_{b_0}^{**}(\vec{f};\Phi)(x)\Big(\frac{|x-(y-z)|}{st}+1\Big)^{b_0}\,dz\;ds\\ \le &\int_0^1 s^{-b_0}\int_{\mathbf{R}^n}|(\Theta^{(s)})_t(z)|\,M_{b_0}^{**}(\vec{f};\Phi)(x)\Big(\frac{|x-y|}{t}+\frac{|z|}{t}+1\Big)^{b_0}\,dz\;ds\\ \le &\,2^{b_0}M_{b_0}^{**}(\vec{f};\Phi)(x)\int_0^1 s^{-b_0}\int_{\mathbf{R}^n}|\Theta^{(s)}(w)|\big(|w|+1\big)^{b_0}\,dw\;ds\\ \le &\,2^{b_0}M_{b_0}^{**}(\vec{f};\Phi)(x)\int_0^1 s^{-b_0}C_0(\Phi,b_0)\,s^{b_0}\,\mathfrak{N}_{b_0}(\varphi)\,ds\,,\end{aligned}$$

where we applied conclusion (14) of Lemma 1. Setting $N=b_0=[b]+1$, we obtain for y in $B(x,t)$ and $\varphi\in\mathcal{F}_N$,

$$\|(\varphi_t*\vec{f})(y)\|_{\ell^2}\le 2^{b_0}C_0(\Phi,b_0)\,M_{b_0}^{**}(\vec{f};\Phi)(x)\,.$$

Taking the supremum over all y in $B(x,t)$, over all $t>0$, and over all φ in $\mathcal{F}_N$, we obtain the pointwise estimate

$$\mathcal{M}_N(\vec{f})(x)\le 2^{b_0}C_0(\Phi,b_0)\,M_{b_0}^{**}(\vec{f};\Phi)(x)\,,\qquad x\in\mathbf{R}^n,$$

where $N=b_0$. This clearly yields (8) if we set $C_4=2^{b_0}C_0(\Phi,b_0)$.

(e) We fix an $\vec{f}\in\mathcal{S}'(\mathbf{R}^n)$ that satisfies $\|\mathcal{M}_N(\vec{f})\|_{L^{p,\infty}}<\infty$ for some fixed positive integer N. To show that $\vec{f}$ is a bounded distribution, we fix a Schwartz function φ and we observe that for some positive constant $c=c_\varphi$, we have that $c\,\varphi$ is an element of $\mathcal{F}_N$ and thus $M_1^*(\vec{f};c\,\varphi)\le\mathcal{M}_N(\vec{f})$. Then

$$c\,\|(\varphi*\vec{f})(x)\|_{\ell^2}\;\le\;\sup_{|z-y|\le 1}\|(c\varphi*\vec{f})(z)\|_{\ell^2}\le M_1^*(\vec{f};c\varphi)(y)\le M_N(\vec{f})(y)$$

for $|y-x|\le 1$. So let $\lambda=c\,\|(\varphi*\vec{f})(x)\|_{\ell^2}$ and then the inequality

$$v_n^{\frac{1}{p}}\tfrac{\lambda}{2}\le\;\tfrac{\lambda}{2}|\{M_N(\vec{f})>\tfrac{\lambda}{2}\}|^{\frac{1}{p}}\le\|M_N(\vec{f})\|_{L^{p,\infty}}<\infty\tag{33}$$

shows that λ is finite and can be controlled by $2\|M_N(\vec{f})\|_{L^{p,\infty}}v_n^{-\frac{1}{p}}$. Here $v_n=|B(0,1)|$ is the volume of the unit ball in $\mathbf{R}^n$. This implies that $\|\varphi*\vec{f}\|_{\ell^2}$ is a bounded function. We conclude that $\vec{f}$ is a bounded distribution. We now show that $\vec{f}$ is an element of $H^{p,\infty}$. We fix a smooth function with compact support θ such that

$$\theta(x)=\begin{cases}1 & \text{if}\quad |x|<1,\\ 0 & \text{if}\quad |x|>2.\end{cases}$$

We observe that the identity

$$
\begin{aligned}
P(x) &= P(x)\theta(x) + \sum_{k=1}^{\infty} \big(\theta(2^{-k}x)P(x) - \theta(2^{-(k-1)}x)P(x)\big) \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \left(\frac{\theta(\,\cdot\,) - \theta(2(\,\cdot\,))}{(2^{-2k} + |\cdot|^2)^{\frac{n+1}{2}}} \right)_{2^k} (x)
\end{aligned}
$$

is valid for all $x \in \mathbf{R}^n$. Setting

$$
\Phi^{(k)}(x) = \big(\theta(x) - \theta(2x)\big) \frac{1}{(2^{-2k} + |x|^2)^{\frac{n+1}{2}}} \,,
$$

we note that for some fixed constant $c_0 = c_0(n,N)$, the functions $c_0\,\theta\,P$ and $c_0\Phi^{(k)}$ lie in $\mathcal{F}_N$ uniformly in $k = 1, 2, 3, \dots$.

Lemma 2. *Let f be a bounded distribution on $\mathbf{R}^n$. Then we have*

$$
(P * f)(x) = ((\theta P) * f)(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} (\Phi^{(k)}_{2^{-k}} * f)(x)
$$

for all $x \in \mathbf{R}^n$, where the series converges in $\mathcal{S}'(\mathbf{R}^n)$.

Combining this observation with the identity for $P(x)$ obtained earlier, we conclude that

$$
\begin{aligned}
\sup_{t>0} \|P_t * \vec{f}\,\|_{\ell^2} &\le \sup_{t>0} \|(\theta P)_t * \vec{f}\,\|_{\ell^2} + \frac{1}{c_0} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sup_{t>0} \sum_{k=1}^{\infty} 2^{-k} \big\|(c_0\Phi^{(k)})_{2^k t} * \vec{f}\,\big\|_{\ell^2} \\
&\le C_5(n,N)\mathcal{M}_N(\vec{f}\,)\,,
\end{aligned}
$$

which proves the required conclusion (9).

We observe that the last estimate also yields the stronger estimate

$$
M_1^*(\vec{f};P)(x) = \sup_{t>0} \sup_{\substack{y\in\mathbf{R}^n \\ |y-x|\le t}} |(P_t * \vec{f}\,)(y)| \le C_5(n,N)\mathcal{M}_N(\vec{f}\,)(x)\,. \tag{34}
$$

It follows that the quasinorm $\|M_1^*(\vec{f};P)\|_{L^{p,\infty}}$ is at most a constant multiple of $\|\mathcal{M}_N(\vec{f}\,)\|_{L^{p,\infty}}$ and thus it is also equivalent to $\|\vec{f}\,\|_{H^{p,\infty}}$.

This concludes the proof of Theorem 1 □

It remains to prove Lemma 2.

Proof of Lemma 2. We begin with the identity

$$
\begin{aligned}
P(x) &= P(x)\theta(x) + \sum_{k=1}^{\infty} \big(\theta(2^{-k}x)P(x) - \theta(2^{-(k-1)}x)P(x)\big) \\
&= P(x)\theta(x) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{\infty} 2^{-k} \left(\frac{\theta(\,\cdot\,) - \theta(2(\,\cdot\,))}{(2^{-2k} + |\cdot|^2)^{\frac{n+1}{2}}} \right)_{2^k} (x)
\end{aligned}
$$

which is valid for all $x \in \mathbf{R}^n$.

Fix a function $\phi \in \mathcal{S}(\mathbf{R}^n)$ whose Fourier transform is equal to 1 in a neighborhood of zero. Then $f = \phi * f + (\delta_0 - \phi) * f$ and we also have $P_t * f = P_t * \phi * f + P_t * (\delta_0 - \phi) * f$. Given a function ψ in $\mathcal{S}(\mathbf{R}^n)$ we need to show that

$$\big\langle (\theta P)_t * \phi * f, \psi \big\rangle + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \big\langle (\Phi^{(k)})_{2^k t} * \phi * f, \psi \big\rangle \to \langle P_t * \phi * f, \psi \rangle$$

and

$$\big\langle (\theta P)_t * (\delta_0 - \phi) * f, \psi \big\rangle + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \big\langle (\Phi^{(k)})_{2^k t} * (\delta_0 - \phi) * f, \psi \big\rangle \to \langle P_t * (\delta_0 - \phi) * f, \psi \rangle$$

as $N \to \infty$.

The first of these claims is equivalent to

$$\big\langle \phi * f, \psi * (\theta P)_t \big\rangle + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \big\langle \phi * f, \psi * (\Phi^{(k)})_{2^k t} \big\rangle \to \langle \phi * f, \psi * P_t \rangle$$

as $N \to \infty$. Here $\phi * f \in L^\infty$ and the actions $\langle \cdot, \cdot \rangle$ are convergent integrals in all three cases. This claim will be a consequence of the Lebesgue dominated convergence theorem since:

$$\psi * (\theta P)_t + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \psi * (\Phi^{(k)})_{2^k t} \to \psi * P_t$$

pointwise (which is also a consequence of the Lebesgue dominated convergence theorem) as $N \to \infty$ and hence the same is true after multiplying by the bounded function $\phi * f$ and also

$$|\phi * f| \ \left| \psi * (\theta P)_t + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \psi * (\Phi^{(k)})_{2^k t} \right| \le |\phi * f| \ \ (|\psi| * P_t) \in L^1(\mathbf{R}^n).$$

We now turn to the corresponding assertion where ϕ is replaced by $\delta - \phi$. Using the Fourier transform, this assertion is equivalent to

$$\big\langle \widehat{f}, \widehat{(\theta P)}_t (1 - \widehat{\phi}) \widehat{\psi} \big\rangle + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \big\langle \widehat{f}, \widehat{(\Phi^{(k)})_{2^k t}} (1 - \widehat{\phi}) \widehat{\psi} \big\rangle \to \langle \widehat{f}, \widehat{P}_t (1 - \widehat{\phi}) \widehat{\psi} \rangle$$

Since $\widehat{f} \in \mathcal{S}'(\mathbf{R}^n)$, this assertion will be a consequence of the fact that

$$\widehat{(\theta P)}_t (1 - \widehat{\phi}) \widehat{\psi} + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \widehat{(\Phi^{(k)})_{2^k t}} (1 - \widehat{\phi}) \widehat{\psi} \to \widehat{P}_t (1 - \widehat{\phi}) \widehat{\psi}$$

in $\mathcal{S}(\mathbf{R}^n)$. It will therefore be sufficient to show that for all multiindices α and β we have

$$\sup_{\xi \in \mathbf{R}^n} \left| \partial_\xi^\alpha \left[\left\{ \widehat{(\theta P)}_t(\xi) + \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{N} 2^{-k} \widehat{(\Phi^{(k)})_{2^k t}}(\xi) - \widehat{P}_t(\xi) \right\} (1 - \widehat{\phi}(\xi)) \widehat{\psi}(\xi) \xi^\beta \right] \right| \to 0$$

The term inside the curly brackets is equal to the Fourier transform of the function $(\theta(2^{-N}x)-1)P_t(x)$, which is

$$\int_{\mathbf{R}^n}(e^{-2\pi t|\xi-\zeta|}-e^{-2\pi t|\xi|})2^{Nn}\widehat{\theta}(2^N\zeta)\,d\zeta$$

since $\widehat{\theta}$ has integral 1. Note that $|\xi|\geq c>0$ since $1-\widehat{\phi}(\xi)=0$ in a neighborhood of zero. Let $\varepsilon_N>0$. First consider the integral

$$\int_{|\xi-\zeta|>\varepsilon_N}(e^{-2\pi t|\xi-\zeta|}-e^{-2\pi t|\xi|})2^{Nn}\widehat{\theta}(2^N\zeta)\,d\zeta\,.$$

Both exponentials are differentiable in this range and differentiation gives

$$\partial_\xi^\gamma\int_{|\xi-\zeta|>\varepsilon_N}(e^{-2\pi t|\xi-\zeta|}-e^{-2\pi t|\xi|})2^{Nn}\widehat{\theta}(2^N\zeta)\,d\zeta$$

$$=\int_{|\xi-\zeta|>\varepsilon_N}\left[Q_\gamma\Big(\tfrac{\xi-\zeta}{|\xi-\zeta|},\tfrac{1}{|\xi-\zeta|}\Big)e^{-2\pi t|\xi-\zeta|}-Q_\gamma\Big(\tfrac{\xi}{|\xi|},\tfrac{1}{|\xi|}\Big)e^{-2\pi t|\xi|}\right]2^{Nn}\widehat{\theta}(2^N\zeta)\,d\zeta\,,$$

where Q_γ is a polynomial of the following $n+1$ variables

$$Q_\gamma\Big(\frac{\xi}{|\xi|},\frac{1}{|\xi|}\Big)=Q_\gamma\Big(\frac{\xi_1}{|\xi|},\dots,\frac{\xi_n}{|\xi|},\frac{1}{|\xi|}\Big)$$

that depends on γ. Note that

$$\left|Q_\gamma\Big(\tfrac{\xi-\zeta}{|\xi-\zeta|},\tfrac{1}{|\xi-\zeta|}\Big)e^{-2\pi t|\xi-\zeta|}-Q_\gamma\Big(\tfrac{\xi}{|\xi|},\tfrac{1}{|\xi|}\Big)e^{-2\pi t|\xi|}\right|\leq C\,|\zeta|\,.$$

Thus the integral is bounded by

$$\int_{|\xi-\zeta|>\varepsilon_N}C\,2^{Nn}|\zeta|\widehat{\theta}(2^N\zeta)\,d\zeta=C'2^{-N}\,,$$

which tends to zero as $N\to\infty$. Now consider the integral

$$\partial_\xi^\gamma\int_{|\xi-\zeta|\leq\varepsilon_N}(e^{-2\pi t|\xi-\zeta|}-e^{-2\pi t|\xi|})2^{Nn}\widehat{\theta}(2^N\zeta)\,d\zeta$$

$$=\partial_\xi^\gamma\int_{|\zeta|\leq\varepsilon_N}(e^{-2\pi t|\zeta|}-e^{-2\pi t|\xi|})2^{Nn}\widehat{\theta}(2^N(\xi-\zeta))\,d\zeta$$

in which $|\xi|\geq c>0$. We obtain another expression which is bounded by

$$C\,\varepsilon_N^n\,2^{N|\gamma|}\leq C\,\varepsilon_N^n\,2^{N|\alpha|}\,,$$

which tends to zero if we pick $\varepsilon_N=2^{-N|\alpha|/n}/N$. Finally

$$\sup_{\xi\in\mathbf{R}^n}\left|\partial_\xi^{\alpha-\gamma}\Big\{(1-\widehat{\phi}(\xi))\widehat{\psi}(\xi)\xi^\beta\Big\}\Big]\right|<\infty$$

and so the claimed conclusion follows by applying Leibniz's rule to the expression on which ∂_ξ^α is acting.

Fix a smooth radial nonnegative compactly supported function θ on $\mathbf{R}^n$ such that $\theta=1$ on the unit ball and vanishing outside the ball of radius 2. Set $\Phi^{(k)}(x)=$

$\big(\theta(x)-\theta(2x)\big)(2^{-2k}+|x|^2)^{-\frac{n+1}{2}}$ for $k\geq 1$. Prove that for all bounded tempered distributions f and for all $t>0$ we have

$$P_t*f=(\theta P)_t*f+\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^{\infty}2^{-k}(\Phi^{(k)})_{2^k t}*f\,,$$

where the series converges in $\mathcal{S}'(\mathbf{R}^n)$. Here $P(x)=\Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}(1+|x|^2)^{\frac{n+1}{2}}$ is the Poisson kernel.

□

4. Properties of $H^{p,\infty}$

The spaces $H^{p,\infty}$ have several properties analogous to those of the classical Hardy spaces H^p. Here we provide a list of these properties and we provide proofs for some of them. The missing proofs can be found in [16].

Theorem 3. *Let $1<p<\infty$. Then we have $L^{p,\infty}=H^{p,\infty}$ and $\|f\|_{L^{p,\infty}}\approx\|f\|_{H^{p,\infty}}$.*

Theorem 4. *(a) For any $0<p<\infty$, every $\vec{f}=\{f_j\}_j$ in $H^{p,\infty}(\mathbf{R}^n,\ell^2)$, and any $\varphi\in\mathcal{S}(\mathbf{R}^n)$ we have*

$$\Big(\sum_j\big|\langle f_j,\varphi\rangle\big|^2\Big)^{1/2}\leq\mathfrak{N}_N(\varphi)\inf_{|z|\leq 1}\mathcal{M}_N(\vec{f})(z)\,,\tag{35}$$

where $N=[\frac{n}{p}]+1$, and consequently there is a constant $C_{n,p}$ such that

$$\Big(\sum_j\big|\langle f_j,\varphi\rangle\big|^2\Big)^{1/2}\leq\mathfrak{N}_N(\varphi)\,C_{n,p}\,\big\|\vec{f}\big\|_{H^{p,\infty}}\,.\tag{36}$$

(b) Let $0<p\leq 1$, $N=[n/p]+1$, and $p<r\leq\infty$. Then there is a constant $C(p,n,r)$ such that for any $\vec{f}\in H^{p,\infty}$ and $\varphi\in\mathcal{S}(\mathbf{R}^n)$ we have

$$\Big\|\Big(\sum_j\big|f_j*\varphi\big|^2\Big)^{1/2}\Big\|_{L^r}\leq C(p,n,r)\mathfrak{N}_N(\varphi)\big\|\vec{f}\big\|_{H^{p,\infty}}\,.\tag{37}$$

(c) For any $x_0\in\mathbf{R}^n$, for all $R>0$, and any $\psi\in\mathcal{S}(\mathbf{R}^n)$ we have

$$\Big(\sum_j\big|\langle f_j,\psi\rangle\big|^2\Big)^{1/2}\leq\mathfrak{N}_N(\psi;x_0,R)\inf_{|z-x_0|\leq R}\mathcal{M}_N(\vec{f})(z)\,.\tag{38}$$

Proposition 2. *Let $0<p<\infty$. The following triangle inequality holds for all f,g in $H^{p,\infty}$:*

$$\|f+g\|_{H^{p,\infty}}^p\leq 2^p(\|f\|_{H^{p,\infty}}^p+\|g\|_{H^{p,\infty}}^p).$$

Moreover, we have

$$\|\{f_j\}\|_{H^{p,\infty}}(\mathbf{R}^n,\ell^2)\approx\sup_{0<|E|<\infty}|E|^{-\frac{1}{r}+\frac{1}{p}}\bigg(\int_E\sup_{t>0}\big\|\{(\varphi_t*f_j)(x)\}_j\big\|_{\ell^2}^r\,dx\bigg)^{\frac{1}{r}}$$

for $0<r<p$.

Proposition 3. *For $0 < p < \infty$, $H^{p,\infty}(\mathbf{R}^n, \ell^2(L))$ are complete quasi-normed spaces.*

Theorem 5. *L^r is not dense in $L^{p,\infty}$, whenever $0 < r \le \infty$ and $0 < p < \infty$.*

The distribution $\delta_1 - \delta_{-1}$ is used to prove the following result.

Theorem 6. *The space of Schwartz functions $\mathcal{S}$ is not dense in $H^{1,\infty}$.*

Interpolation is a powerful tool that yields results in the theory of weak Hardy spaces. The following lemma will be useful in the proof of Theorem 9.

Lemma 3. *([16]) Let $0 < p_1 < p < p_2 < \infty$. Given $\vec{F} = \{f_k\}_{k=1}^L \in H^{p,\infty}(\mathbf{R}^n, \ell^2(L))$ and $\alpha > 0$, then there exists $\vec{G} = \{g^k\}_{k=1}^L$ and $\vec{B} = \{b^k\}_{k=1}^L$ such that $\vec{F} = \vec{G} + \vec{B}$ and*

$$\|\vec{B}\|^{p_1}_{H^{p_1}(\mathbf{R}^n,\ell^2(L))} \le C\alpha^{p_1-p}\|F\|^p_{H^{p,\infty}(\mathbf{R}^n,\ell^2(L))}$$

and

$$\|\vec{G}\|^{p_2}_{H^{p_2}(\mathbf{R}^n,\ell^2(L))} \le C\alpha^{p_2-p}\|\vec{F}\|^p_{H^{p,\infty}(\mathbf{R}^n,\ell^2(L))}$$

where $C = C(p_1, p_2, p, n)$, in particular is independent of L.

Next, we will discuss the cancellation of Schwartz functions in $H^{p,\infty}$ for $p \le 1$. We denote by $[a]$ the integer part of a real number a. We have the following result:

Theorem 7. *If $f \in H^p(\mathbf{R}^n)$, then $\int_{\mathbf{R}^n} x^\gamma f(x)\,dx = 0$ for $|\gamma| \le [\frac{n}{p} - n]$ if these integrals converge absolutely. If $f \in H^{p,\infty}(\mathbf{R}^n)$, then $\int_{\mathbf{R}^n} x^\gamma f(x)\,dx = 0$ for $|\gamma| \le -[n - \frac{n}{p}] - 1$ if these integrals converge absolutely.*

Proof. For $f \in H^p$, we have $|\eta_t * f(x)| \le \frac{C}{t^{\frac{n}{p}}}\|f\|_{H^p}$ and $\|\eta_t * f\|_{L^p} \le C\|f\|_{H^p}$, where $\eta \in \mathcal{S}$ with $1 \ge \widehat{\eta} \ge 0$, $\widehat{\eta}(\xi) = 1$ for $|\xi| \le 1$ and 0 for $|\xi| \ge 2$. Therefore $\|\eta_t * f\|_{L^1} \le Ct^{n-\frac{n}{p}}\|f\|_{H^p}$. For each fixed $|\xi|$ we can take $t = \frac{1}{2|\xi|}$, which would imply that $|\widehat{f}(\xi)| \le C|\xi|^{\frac{n}{p}-n}\|f\|_{H^p}$. All Schwartz functions ψ with $\int_{\mathbf{R}^n} x^\gamma \psi = 0$ for all γ are dense in H^p. For such a ψ we have $\lim_{|\xi|\to 0} \widehat{\psi}(\xi)/|\xi|^{\frac{n}{p}-n} = 0$. By the density we mentioned we know that for all f in H^p we must have $\lim_{|\xi|\to 0} \widehat{f}(\xi)/|\xi|^{\frac{n}{p}-n} = 0$. And this limit gives us that $\partial^\gamma \widehat{f}(0) = 0$ for $|\gamma| \le [\frac{n}{p} - n]$, i.e. $\int_{\mathbf{R}^n} x^\gamma f(x)dx = 0$, once we notice that $\partial^\gamma \widehat{f}(\xi)$ are well-defined and continuous.

Next we will prove the corresponding result for $H^{p,\infty}$. By lemma 3 we know that if $f \in H^{p,\infty}$, then $f = h + g$ with $h \in H^{p_1}$ and $g \in H^{p_2}$, where $p_1 < p < p_2$. Moreover h and g have the same integrability as f since they're truncation of f. We have $\lim_{|\xi|\to 0} \widehat{h}(\xi)/|\xi|^{\frac{n}{p_1}-n} = 0$ and $\lim_{|\xi|\to 0} \widehat{g}(\xi)/|\xi|^{\frac{n}{p_2}-n} = 0$, which implies $\lim_{|\xi|\to 0} \widehat{f}(\xi)/|\xi|^{\frac{n}{p_2}-n} = 0$ for any $p_2 > p$. This result will give us the same cancellation for f as that in H^p if $\frac{n}{p} - n$ is not an integer but one degree less if it's an integer, namely $\int_{\mathbf{R}^n} x^\gamma f(x)\,dx = 0$ for all $|\gamma| \le -[n - \frac{n}{p}] - 1$. □

A natural question is that if the number $-[n - \frac{n}{p}] - 1$ is sharp. The fact that $L^1 \subset H^{1,\infty}$, which is the result of weak type $(1, 1)$ boundedness of Hardy-Littlewood maximal function, suggests that this number is sharp. Indeed this is the case and we give a precise proof below.

We can get the converse of the previous theorem, i.e. every Schwartz function satisfying the cancellation of the theorem must be in the corresponding space. We need the following lemma which can be found in Appendix K.2 of [12].

Lemma 4. *Let $M, N > 0$ and L a nonnegative integer. Suppose that φ_μ and φ_ν are functions on $\mathbf{R}^n$ that satisfy*

$$|\partial_x^\alpha \varphi_\mu(x)| \le \frac{A_\alpha 2^{\mu n} 2^{\mu L}}{(1+2^\mu|x-x_\mu|)^M} \qquad \text{for all } |\alpha| = L,$$

$$|\varphi_\nu(x)| \le \frac{B 2^{\nu n}}{(1+2^\nu|x-x_\nu|)^N},$$

for some A_α and B positive, and $\int_{\mathbf{R}^n} \varphi_\nu(x) x^\beta dx = 0$ for all $|\beta| \le L-1$, where the last condition is supposed to be vacuous when $L = 0$. Suppose that $N > M + L + n$ and $\nu \ge \mu$. Then we have

$$\left| \int_{\mathbf{R}^n} \varphi_\mu \varphi_\nu \, dx \right| \le C \frac{2^{\mu n} 2^{-(\nu-\mu)L}}{(1+2^\mu|x_\mu - x_\nu|)^M}.$$

Theorem 8. *Any $f \in \mathcal{S}$ with $\int_{\mathbf{R}^n} x^\gamma f(x)dx = 0$ for $|\gamma| \le [\frac{n}{p} - n]$ lies in $H^p(\mathbf{R}^n)$. Any $f \in \mathcal{S}$ with $\int_{\mathbf{R}^n} x^\gamma f(x)dx = 0$ for all $|\gamma| \le -[n - \frac{n}{p}] - 1$ lies in $H^{p,\infty}(\mathbf{R}^n)$.*

Proof. We want to estimate

$$f^+(x) = \sup_{t>0} |(\psi_t * f)(x)| = \sup_{t>0} \left| \int_{\mathbf{R}^n} f(y)\psi_t(x-y)dy \right|.$$

Take $\psi_t = \varphi_\mu$, $f = \varphi_\nu$, $L = [\frac{n}{p} - n] + 1$ for the first case and $-[n - \frac{n}{p}]$ for the second, $\nu = 0$. The condition $\mu \le \nu$ forces $t^{-1} = 2^\mu \le 1$. So

$$\begin{aligned} \sup_{t\ge 1} \left| \int_{\mathbf{R}^n} f(y)\psi_t(x-y)dy \right| &\le C \sup_{t\ge 1} \frac{t^{-n}t^{-L}}{(1+t^{-1}|x|)^M} \\ &= C \sup_{t\ge 1} \frac{1}{(t+|x|)^{n+L}} \frac{1}{(1+t^{-1}|x|)^{M-n-L}} \\ &\le C \frac{1}{(1+|x|)^{n+L}}. \end{aligned}$$

If we take $f = \varphi_\mu$, $\psi_t = \varphi_\nu$, $L_0 = 0$, $\mu = 0$, then $t^{-1} = 2^\nu \ge 1$. We have

$$\sup_{0<t\le 1} \left| \int_{\mathbf{R}^n} f(y)\psi_t(x-y)dy \right| \le C \sup_{0<t\le 1} \frac{2^{\nu L_0}}{(1+|x|)^M} \le \frac{C}{(1+|x|)^{n+L}}.$$

Thus $f^+(x)$ is controlled by $C(1+|x|)^{-[\frac{n}{p}]-1}$, which is in L^p, while the second kind of cancellation implies that $f^+(x)$ is in $H^{p,\infty}$. □

Next we will get a corollary of these two theorems to characterize the class of Schwartz functions

$$\mathcal{S}_k = \left\{ f \in \mathcal{S}(\mathbf{R}^n) : \int_{\mathbf{R}^n} x^\gamma f(x)dx = 0, \ |\gamma| \le k \right\}$$

in terms of the decay of the corresponding smooth maximal functions of their elements.

Corollary 1. *(i) For a Schwartz function f the following equivalence is valid:*

$$f \in \mathcal{S}_k \Leftrightarrow f^+(x) \leq \frac{C}{(1+|x|)^{n+k+1}};$$

(ii) For a Schwartz function f if $f^+(x) \leq \frac{C}{(1+|x|)^{n+k+\epsilon}}$ for some $\epsilon > 0$, then $f^+(x) \leq \frac{C}{(1+|x|)^{n+k+1}}$.

Proof. (i) The forward direction comes from the calculation of Theorem 8. The backwards direction is a result of Theorem 7 since such a function f lies in some H^p with $k = [\frac{n}{p} - n]$.

(ii) It's easy to see that this f is in H^p for $p = \frac{n}{n+k}$, therefore $f \in \mathcal{S}_k$ by Theorem 7, and the conclusion follows by (i). □

A result similar to that $L^p \cap \mathcal{S} = L^q \cap \mathcal{S}$ for $p, q \geq 1$ will be revealed by next corollary.

Corollary 2. *(i) $H^p \cap \mathcal{S} = H^q \cap \mathcal{S}$ for $p, q \in (\frac{n}{n+k+1}, \frac{n}{n+k}]$, $k \in \mathbf{N}$.*
(ii) $H^{p,\infty} \cap \mathcal{S} = H^{q,\infty} \cap \mathcal{S}$ for $p, q \in [\frac{n}{n+k+1}, \frac{n}{n+k})$, $k \in \mathbf{N}$
(iii) $H^{p,\infty} \cap \mathcal{S} = H^p \cap \mathcal{S}$ for $p \neq \frac{n}{n+k}$.
(iv) The statement in (i) fails for $p = \frac{n}{n+k+1}$ and (ii) fails for $p = \frac{n}{n+k}$.
(v) For all $f \in \mathcal{S}_k := H^p \cap \mathcal{S}$, $p \in (\frac{n}{n+k+1}, \frac{n}{n+k}]$ and all $x \in \mathbf{R}^n$, the best estimate for $f^+ = \sup_{t>0} |(f * \varphi_t)|$ is $f^+(x) \leq \frac{C}{(1+|x|)^{n+k+1}}$.

Proof. (i) Suppose that $f \in H^p \cap \mathcal{S}$. Then $\int_{\mathbf{R}^n} x^\gamma f(x) dx = 0$ for all $|\gamma| \leq [\frac{n}{p} - n]$. This implies that $f^+(x) \leq C(1+|x|)^{-([\frac{n}{p}-n]+1+n)}$ and in turn this implies that $f \in H^q$ for $[\frac{n}{q} - n] \leq [\frac{n}{p} - n]$. This implies the required conclusion since $[\frac{n}{p}] = [\frac{n}{q}]$ for $p, q \in (\frac{n}{n+k+1}, \frac{n}{n+k}]$, whenever $k \geq 0$.

(ii) The proof is similar to (i) but notice that a given $f \in H^{p,\infty}$ satisfies the cancellation condition that $\int_{\mathbf{R}^n} x^\gamma f(x)\, dx = 0$ for $|\gamma| \leq -[n - \frac{n}{p}] - 1$.

(iii) This is a consequence of the fact that $[r] = -[-r] - 1$ for r is not an integer, Theorem 7 and Theorem 8.

(iv) Let's consider only the H^p case since it's equivalent to the $H^{p,\infty}$ case. We can consider $g \in \mathcal{S}_k$ such that $g \notin \mathcal{S}_{k+1}$ (e.g. $g(x) = x^k e^{-|x|^2}$). This g is in H^p for $p \in (\frac{n}{n+k+1}, \frac{n}{n+k}]$ but not $p = \frac{n}{n+k+1}$, otherwise $g \in \mathcal{S}_{k+1}$ by Theorem 7.

(v) If we had $f^+(x) \leq \frac{C}{(1+|x|)^{n+k+1+\epsilon}}$ for some $\epsilon > 0$ and all $x \in \mathbf{R}^n$, then it follows from Corollary 1 that f would belong to $H^{\frac{n}{n+k+1}}$, which is not true by (iv). □

The existence of a function g in (iv) shows that we cannot improve the cancellation of Theorem 7 for H^p and $H^{p,\infty}$.

5. Square function characterization of $H^{p,\infty}$

In this section we outline the proof of a new characterization of weak Hardy spaces in terms of Littlewood–Paley square functions; details can be found in [16]. We begin by stating an interpolation theorem and a consequence of it first.

Theorem 9. *Let J and L be positive integers and let $0 < p_1 < p < p_2 < \infty$, moreover $p_1 \le 1$.*
(a) Let T be a sublinear operator defined on $H^{p_1}(\mathbf{R}^n, \ell^2(L)) + H^{p_2}(\mathbf{R}^n, \ell^2(L))$. Assume that T maps $H^{p_1}(\mathbf{R}^n, \ell^2(L))$ to $H^{p_1}(\mathbf{R}^n, \ell^2(J))$ with constant A_1 and $H^{p_2}(\mathbf{R}^n, \ell^2(L))$ to $H^{p_2}(\mathbf{R}^n, \ell^2(J))$ with constant A_2. Then there exists a constant $c_{p_1,p_2,p,n}$ independent of J and L such that

$$\|T(\vec{F})\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2(J))} \le c_{p_1,p_2,p,n}\, A_1^{\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}} A_2^{\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}} \|\vec{F}\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2(L))}$$

for $\vec{F} \in H^{p,\infty}(\mathbf{R}^n, \ell^2(L))$.

(b) Suppose that T is a sublinear operator defined on $H^{p_1}(\mathbf{R}^n, \ell^2(L)) + H^{p_2}(\mathbf{R}^n, \ell^2(L))$. Assume that T maps $H^{p_1}(\mathbf{R}^n, \ell^2(L))$ to $L^{p_1}(\mathbf{R}^n, \ell^2(J))$ with constant A_1 and also maps $H^{p_2}(\mathbf{R}^n, \ell^2(L))$ to $L^{p_2}(\mathbf{R}^n, \ell^2(J))$ with constant A_2. Then there exists a constant C independent of J and L such that

$$\|T(\vec{F})\|_{L^{p,\infty}(\mathbf{R}^n,\ell^2(J))} \le c_{p_1,p_2,p,n}\, A_1^{\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}} A_2^{\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}} \|\vec{F}\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2(L))}$$

for all distributions $\vec{F} \in H^{p,\infty}(\mathbf{R}^n, \ell^2(L))$.

Corollary 3. *Let $0 < p < \infty$ and suppose that $\{K_j(x)\}_{j=1}^L$ is a family of kernels defined on $\mathbf{R}^n \setminus \{0\}$ satisfying*

$$\sum_{j=1}^{L} |\partial^\alpha K_j(x)| \le A|x|^{-n-|\alpha|} < \infty$$

for all $|\alpha| \le \max\{[n/p]+2, n+1\}$ and

$$\sup_{\xi\in\mathbf{R}^n} \sum_{j=1}^{L} |\widehat{K_j}(\xi)| \le B < \infty\,.$$

Then for some $0 < p$ there exists a constant $C_{n,p}$ independent of L such that

$$\Big\| \sum_{j=1}^{L} K_j * f_j \Big\|_{H^{p,\infty}(\mathbf{R}^n)} \le C_{n,p}(A+B) \|\{f_j\}_{j=1}^L\|_{H^{p,\infty}(\mathbf{R}^n,\ell^2(L))}. \tag{39}$$

We fix a radial function $\Psi \in \mathcal{S}(\mathbf{R}^n)$ such that $\widehat{\Psi}$ is nonnegative, supported in the annulus $\frac{1}{2} + \frac{1}{10} \le |\xi| \le 2 - \frac{1}{10}$, and satisfies

$$\sum_{j\in\mathbf{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 \tag{40}$$

for all $\xi \neq 0$. We define the related Littlewood–Paley operators Δ_j by

$$\Delta_j(f) = \Delta_j^{\Psi}(f) = \Psi_{2^{-j}} * f\,. \tag{41}$$

We also define the function Φ by $\widehat{\Phi}(\xi) = \sum_{j\leq 0} \widehat{\Psi}(2^{-j}\xi)$ for $\xi \neq 0$ and $\widehat{\Phi}(0) = 1$. We now provide a sketch of the proof of Theorem 2.

Proof. Choose $f \in H^{p,\infty}$ and denote $f_M = \sum_{|j|\leq M} \Delta_j(f) = \Phi_{2^{-M}} * f - \Phi_{2^M} * f$ and $S(f) = (\sum_{|j|\leq M} |\Delta_j(f)|^2)^{\frac{1}{2}}$. Then by Theorem 9 it follows that S maps $H^{p,\infty}$ to $L^{p,\infty}$ bounded for $p \in (p_1, p_2)$, so

$$\|S(f)\|_{L^{p,\infty}} \leq C\|f\|_{H^{p,\infty}}.$$

Applying Fatou's lemma for $L^{p,\infty}$ spaces we have

$$\|(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2)^{\frac{1}{2}}\|_{L^{p,\infty}} \leq \liminf_{M\to\infty} \|(\sum_{|j|\leq M} |\Delta_j(f)|^2)^{\frac{1}{2}}\|_{L^{p,\infty}} \leq C\|f\|_{H^{p,\infty}}.$$

Now let's assume we have a distribution $f \in \mathcal{S}'$ such that

$$\|(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2)^{\frac{1}{2}}\|_{L^{p,\infty}} < \infty.$$

By lemma 6.5.3 of [13] and Proposition 1, we can show that $\{\Delta_j * f\}_{j\in\mathbf{Z}} \in H^{p,\infty}(\mathbf{R}^n, l^2)$ with that

$$\big\|\sup_{t>0}(\sum_{j\in\mathbf{Z}} |\varphi_t * \Delta_j(f)|^2)^{\frac{1}{2}}\big\|_{L^{p,\infty}} \leq C_p'\big\|(\sum_{j\in\mathbf{Z}}(|\Delta_j(f)|)^2)^{\frac{1}{2}}\big\|_{L^{p,\infty}}.$$

Let $\widehat{\eta}(\xi) = \widehat{\Psi}(\xi/2) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi)$, then by Corollary 3

$$\big\|\sum_{|j|\leq M} \Delta_j(f)\big\|_{H^{p,\infty}} \leq C\,\big\|(\sum_{j\in\mathbf{Z}}(|\Delta_j(f)|)^2)^{\frac{1}{2}}\big\|_{L^{p,\infty}}.$$

So $\{\sum_{|j|\leq M} \Delta_j(f)\}_M$ is a bounded sequence in $H^{p,\infty}$ uniformly in M. Then we use the following lemma contained in [16].

Lemma 5. *If $\{f_j\}$ is bounded by B in $H^{p,\infty}$ (or H^p), then there exists a subsequence $\{f_{j_k}\}$ such that $f_{j_k} \to f$ in $\mathcal{S}'$ for some f in $H^{p,\infty}$ (or H^p) with $\|f\|_{H^{p,\infty}} \leq B$ (or $\|f\|_{H^p} \leq B$).*

By the lemma $\sum_{|j|\leq M_k} \Delta_j(f) \to g$ in $\mathcal{S}'$ with $\|g\|_{H^{p,\infty}} \leq C\|(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2)^{\frac{1}{2}}\|_{L^{p,\infty}}$. Moreover there is a unique polynomial Q such that $g = f - Q$. □

Different functions Ψ provide spaces with equivalent norms. This is easily seen by the above characterization.

Corollary 4. *The definition of space*

$$F^{p,\infty} := \big\{f \in \mathcal{S}' :\ \big\|\big(\sum_{j\in\mathbf{Z}} |\Delta_j(f)|^2\big)^{\frac{1}{2}}\big\|_{L^{p,\infty}} < \infty\big\}$$

is independent of the choice of Ψ.

This is a consequence of Theorem 2 and can also be a consequence of Proposition 1. The argument also applies to

$$F_p^{\alpha,q} = \big\{ f \in \mathcal{S}' : \ \big\| \big(\sum_{j\in\mathbf{Z}} |2^{j\alpha}\Delta_j(f)|^q \big)^{\frac{1}{q}} \big\|_{L^p} < \infty \big\},$$

so we are allowed to define more general spaces generalizing $H^{p,\infty}$

$$F_{p,\infty}^{\alpha,q} = \big\{ f \in \mathcal{S}' : \ \big\| \big(\sum_{j\in\mathbf{Z}} |2^{j\alpha}\Delta_j(f)|^q \big)^{\frac{1}{q}} \big\|_{L^{p,\infty}} < \infty \big\}.$$

The square function characterization of weak H^p has useful applications; for instance it was used in [14] to obtain weak type endpoint estimates for multilinear paraproducts.

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